AP Physics C: Mechanics
Unit 6 – Torque and Rotational Dynamics
Assessment

Name: ____ Period: __________

Summary

- The **instantaneous angular velocity ω** of a particle rotating in a circle or of a rigid body rotating about a fixed axis is:

$$\omega = \frac{d\theta}{dt}$$

where ω is in rad/s or s^{-1}.

- The **instantaneous angular acceleration α** of a rotating body is given by:

$$\alpha = \frac{d\omega}{dt}$$

where α is in rad/s^2 or s^{-2}.

- When a rigid body rotates about a fixed axis, every part of the body has the same angular acceleration and the same angular velocity. However, different parts of the body have different linear velocities and accelerations.
- If a particle or body undergoes rotational motion about a fixed axis under constant angular acceleration **α**, the equations of kinematics apply by simply converting the linear analogs:

$$\omega_f = \omega_i + \alpha t$$

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$$

$$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$$

$$\theta_f - \theta_i = \frac{1}{2}(\omega_f + \omega_i)t$$

- When a rigid body rotates about a fixed axis, the angular velocity and angular acceleration are related to the linear velocity and linear acceleration through the relationships:

$$v = r\omega$$

$$a = r\alpha$$

- The **moment of inertia** *I* of a **system of particles** is given by:

$$I = \sum m_i r_i^2$$

- The **moment of inertia** *I* of a **rigid body** is given by:

$$I = \int r^2 dm$$

where *r* is the distance from the mass element *dm* to the axis of rotation.

- If a rigid body rotates about a fixed axis with an angular velocity **ω**, its **rotational kinetic energy** can be written as:

$$T_R = \frac{1}{2} I\omega^2$$

- The **net work** done by external forces in rotating a rigid body about a fixed axis equals the change in the rotational kinetic energy of the body:

$$W = \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_i^2$$

This is the **Work-Energy Theorem** applied to rotational motion.

- The **total kinetic energy** *T* of a rigid body that is rolling, such as on a rough surface so that there is no slippage, equals the rotational kinetic energy about its center of mass plus the translational kinetic energy of its center of mass:

$$T = \frac{1}{2} I_c\omega^2 + \frac{1}{2} M v_c^2$$

Torque and Angular Momentum

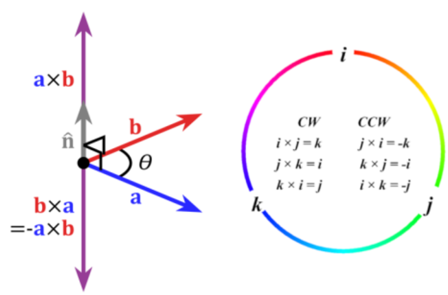
- The **torque, $\boldsymbol{\tau}$**, due to a force ***F*** about an origin (pivot) in an inertial frame is a vector and defined to be:

$$\boldsymbol{\tau = r \; X \; F}$$

This is a **vector cross product** (or **vector product**), which has a magnitude:

$$\boldsymbol{\tau = |r||F| sin\theta}$$

The direction is determined by the rules established for unit vectors:



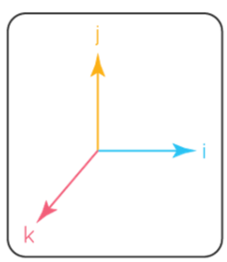
Matrices should be used to solve a vector cross product if the information is given in unit vector form:

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \qquad \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{c} = \vec{a} \times \vec{b}$$

$$\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\vec{c} = \hat{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$



$$\boxed{\vec{c} = \hat{i}\,|a_2b_3 - a_3b_2| - \hat{j}\,|a_1b_3 - a_3b_1| + \hat{k}\,|a_1b_2 - a_2b_1|}$$

- In its simplest form (when the force is perpendicular), the magnitude of the torque τ associated with a force **F** acting upon a body is:

$$\boldsymbol{\tau} = \boldsymbol{rF}$$

- According to Newton's Second Law, forces and torques in a system in equilibrium (where linear and rotational accelerations are zero) are summed. Torque is considered to by positive when counterclockwise and negative when counterclockwise:

$$\sum \boldsymbol{\tau} = \boldsymbol{\tau_1} + \boldsymbol{\tau_2} + \cdots = \mathbf{0}$$

- If a rigid body free to rotate about a fixed axis has a net external torque acting on it, the body will undergo an acceleration **α** described by:

$$\boldsymbol{\tau_{net}} = \boldsymbol{I\alpha}$$

- The rate at which work is being done by external forces – the **power** delivered – is given by:

$$P = \tau\omega$$

- The **angular momentum *L*** of a particle of linear momentum ***p*** = *m**v*** is given by:

$$\boldsymbol{L} = \boldsymbol{r}\ \boldsymbol{X}\ \boldsymbol{p} = \boldsymbol{mr}\ \boldsymbol{X}\ \boldsymbol{v}$$

  If φ is the angle between ***r*** and *p*, the magnitude of the angular momentum vector cross product is:

$$L = mvrsin\varphi$$

  with the direction following the same rules as above.

- The **net external torque** acting on a particle or rigid body is equal to the time rate of change of its angular momentum:

$$\sum \boldsymbol{\tau}_{ext} = \frac{d\boldsymbol{L}}{dt}$$

- The *z component* of angular momentum of a rigid body rotating about a fixed axis (the z axis) is given by:

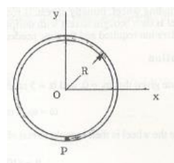
$$\boldsymbol{L_z} = \boldsymbol{I\omega}$$

- If the net external torque acting on a system is zero, the total angular momentum of the system is constant. This yields the Conservation of Angular Momentum (analogous to the Conservation of Linear Momentum) and is given by:

$$I_i\omega_i = I_f\omega_f = constant$$

Concepts

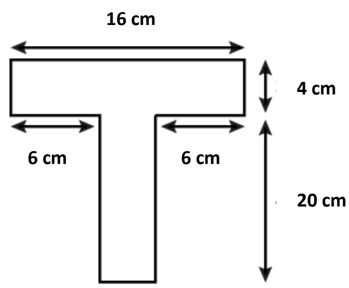
1) A wheel is rotating counterclockwise in the xy plane. What is the direction of its angular velocity? If the angular velocity is decreasing in time, what is the direction of the angular acceleration?
2) Are the kinematic expressions for rotational dynamics equally valid if angular displacement is given in degrees instead of radians? Explain your reasoning.
3) Mr. Webber is listening to his favorite Barry Manilow record on an old-fashioned turntable that rotates at a constant rate of 45 rev/min.
    a. What is the magnitude of its angular velocity in rad/s?
    b. What is the angular acceleration?
4) Consider a simple hula hoop of uniform mass distribution. In two separate experiments, the hoop is rotated from rest to an angular velocity ω. In one experiment, the rotation occurs about the z-axis; in the other experiment, the rotation occurs about an axis parallel to z through P. Which rotation requires more work?



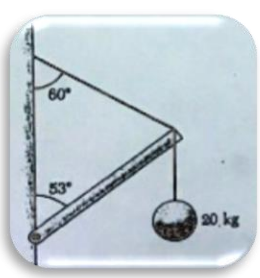
5) Suppose that only two external forces act on a rigid body, and both forces are equal in magnitude but opposite in direction. Under what conditions will the body rotate?
6) A grinding wheel, often used in metal shops, is rotated for 8 s with constant angular acceleration $\alpha = 5$ rad/s^2. It is then brought to rest in 10 revolutions. Determine the negative acceleration required and the time needed to bring the wheel to a stop.
7) Is it possible to calculate the torque acting on a rigid body without specifying an origin? Is torque a quantity independent of the location of the origin? Explain your reasoning.
8) Consider the following expressions. Is the result a scalar, vector, neither, or undefined? Explain your reasoning.
    a. **A**·(**B** X **C**)
    b. **(A·B**) X **C**
9) Under what condition does the moment arm equal the position vector ***r***?
10) Can a particle moving in a straight line have a nonzero angular momentum? Explain.
11) Suppose that in a given system the velocity vector ***v*** is completely defined. What can you conclude about the direction of the angular momentum vector ***L*** with respect to the direction of motion?
12) A baseball is thrown in such a manner that it does not spin about any axis. Does this mean that the angular momentum is zero about some arbitrary origin? Explain.
13) When a cylinder rolls on a horizontal surface experiencing rolling friction only, are there any points on the cylinder that have only a vertical component of velocity at some instant? If so, where. Explain your reasoning.
14) Can a body be in equilibrium if only one external force or one external torque acts upon it. Explain.
15) Can a body be in equilibrium if it is in motion? Explain.
16) Can a body be in equilibrium if there are only counterclockwise torques acting on it?
17) There are two crates, one tall and one short, of equal mass. They are placed next to each other, without touching, on an inclined plane. As the angle θ of the inclined plane is increased, which crate will topple first? Why?
18) *"Lift with your legs and not with your back."* Why do health and safety experts recommend that you should straighten your back as much as possible and use your legs when lifting a heavy object rather than bending over and lifting the object with your arms?

Problems

19) Consider a uniform solid disk and a uniform hoop rolling down an incline.
    a. Determine the acceleration of the center of mass of each.
    b. What is the minimum coefficient of friction required to maintain pure rolling motion for the disk?
20) Given |**A** x **B**| = **A**·**B**. what is the angle between **A** and **B**? Show you reasoning.
21) A non-grooved yo-yo. A string is wound around a disk of thickness *z*, radius *R*, and mass *M*. The string is tied to a horizontal support and the disk is released from rest from a height *h*. As the disk descends
    a. find the tension ***T*** in the string as a function of the disk's weight.
    b. find the acceleration of the center of mass as a function of ***g***.
    c. find the velocity, ***v***, of the center of mass as a function of ***g*** and *h*. *Hint: Energy considerations would be the best way to find velocity.*
22) A flat plate in the shape of a letter T is cut with the dimensions shown below. Locate the center of gravity.



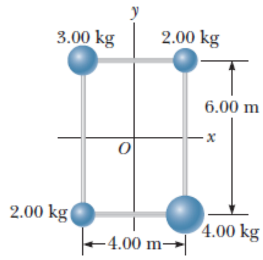
23) A beam of uniform mass density has a mass of 10 kg and a length of 4 m. It supports a 20 kg mass at one end in the configuration shown below.
    a. Draw a free-body diagram of the beam.
    b. Determine the tension ***T*** in the supporting wire.



24) A 500 N, 15 m ladder of uniform mass distribution rests against a frictionless wall, making an angle of 60^{o} with the horizontal. An 800 N firefighter is 4 m from the bottom of the ladder.
    a. Find the horizontal and vertical forces that that Earth exerts on the base of the ladder.
    b. If the ladder is just on the verge of slipping when the firefighter has climber the ladder to 9 m, what is the coefficient of static friction between the ladder and the ground?

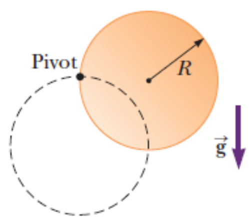
25) Four particles are connected by light, rigid rods, as shown below. If the system rotates in the *xy* plane about the *z* axis with an angular velocity of 6 rad/s, calculate

a. the moment of inertia of the system about the z axis.
b. the kinetic energy of the system.



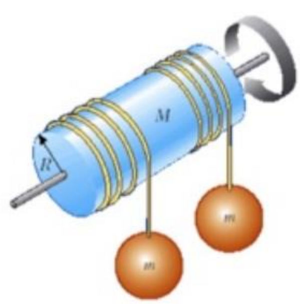
26) Consider a disk of uniform mass density with radius *R* and mass *M* that is free to rotate about a pivot of minimal friction, as shown below. If the disk is released from its initial position (solid line)

a. what is the velocity of the center of mass of the disk as it passes through the lower position (dashed line)?
b. what is the speed of the lowest point of the disk as it passes through the lower position?



27) A solid cylinder of uniform mass distribution, mass *M*, and radius *R*, rotates on a horizontal axle that may be considered to be frictionless. Two equal masses, *m*, hang from light cords (read: "massless") that are wrapped around the cylinder, as shown below. The system is released from rest.

a. Find the tension ***T*** in each cord.
b. Find the acceleration of each mass.
c. Find the angular velocity of the cylinder after the masses have descended a distance *h*.



28) A light nylon cord of length 4 m is wound around a cylindrical spool of mass 1 kg and radius 0.5 m. This cylindrical spool, initially at rest, is placed on a frictionless axle. The cord is then pulled from the spool with a constant acceleration of 2.5 m/s^2.

a. How much work has been done on the spool when it reaches an angular speed of ω = 8 rad/s?
b. How long will it take the spool to reach an angular speed of 8 rad/s?
c. Verify (with physics!) that there enough nylon cord to enable the spool to reach this angular speed of 8 rad/s.

29) A uniform rod of length L and uniform mass M is pivoted through a horizontal, frictionless pin at one end. The rod is released from a vertical position and allow to fall. When the rod has rotated 90 degrees and is horizontal, find

a. the angular velocity of the rod.
b. the angular acceleration of the rod.
c. the x- and y-components of the acceleration of its center of mass.
d. the components of the reaction force (Newton's Third Law) at the pivot.